REL SOUTHEAST
Regional Educational Laboratory
At Florida State University

Professional Learning Community:

IMPROVING MATHEMATICAL PROBLEM SOLVING

FOR STUDENTS IN GRADES 4 THROUGH 8

PARTICIPANT'S ACTIVITIES

Contents

Handout 1A.1: Overview of *How-to Steps*

Recommendation 3: Teach students how to use visual representations

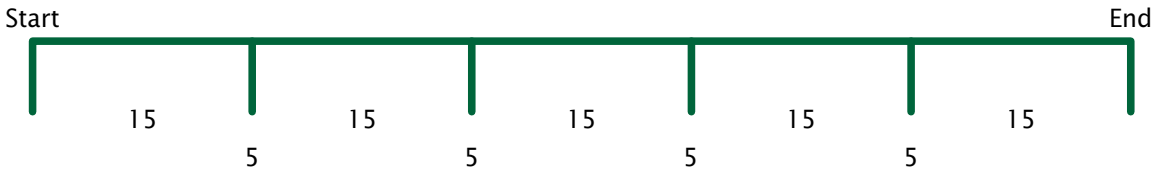
***How-to* Step 1**	***How-to* Step 2**	***How-to* Step 3**
Select visual representations that are appropriate for students and the problems they are solving. • Select appropriate visuals for problem structure • Use visuals consistently • Determine if the visual representations work well for some problems, but not necessarily for all problems • Provide time for students to practice visuals	Use think-alouds and discussions to teach students how to represent problems visually. • Think aloud to connect the problem information to the visual representation • Focus on the thinking rather than only on procedural aspects • Discuss linking problem information to the visual • Discuss important problem information • Identify and ignore irrelevant information • Discuss what is unknown • Promote discussion by asking guiding questions	Show students how to convert the visually represented information into mathematical notation. • Connect the quantities and relationships in the visual to the mathematical notation • Illustrate the connections explicitly, which can be accomplished through think-alouds

Handout 1A.2

Explore Visual Representations – Schematic Diagrams, Tables, and Strip Diagrams

John recently participated in a 5-mile run. He usually runs 2 miles in 30 minutes. Because of an ankle injury, John had to take a 5-minute break after every mile. At each break, he drank 4 ounces of water. How much time did it take him to complete the 5-mile run?

Schematic Diagram

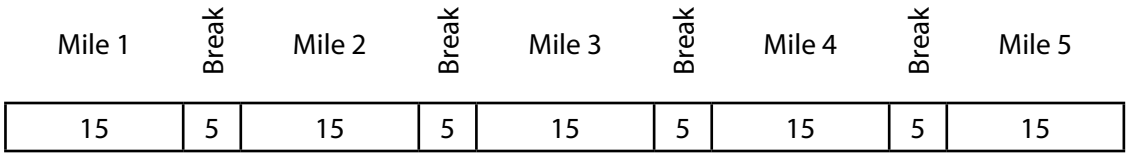


x = total time
$x = (5 \cdot 15) + (4 \cdot 5)$
$x = 95$ minutes

Table

Miles	Mile minutes	Break minutes	Total Time in minutes
1	15	5	20
2	15	5	40
3	15	5	60
4	15	5	80
5	15	End	95

Strip Diagram



$4\ (15 + 5) + 15 = x$

Handout 1A.3

Video Viewing Guide – Introduction to Recommendation 3

Directions: In this video, John Woodward provides a brief overview of Recommendation 3, including a description of the three *How-to* steps for carrying out the recommendation. As you watch the video, take notes on the important information about each *How-to* step.

Recommendation 1: Teach students how to use visual representations.

***How-to* Steps**	**Notes**
***How-to* Step 1:** Select visual representations that are appropriate for students and the problems they are solving.	
***How-to* Step 2:** Use think-alouds and discussions to teach students how to represent problems visually.	

Handout 1A.3 (Continued)

How-to Steps	Notes
***How-to* Step 3:** Show students how to convert the visually represented information into mathematical notation.	

Handout 1A.4
Questions to Answer Before Selecting Visual Representations

Directions: In this video, John Woodward provides a brief overview of Recommendation 3, including a description of the three *How-to* steps for carrying out the recommendation. As you watch the video, take notes on the important information about each *How-to* step.

Questions to guide thinking for selecting a visual:

1. What is the problem about (rate, proportion/ratio, money spent, etc.)?
2. What is unknown (e.g., start amount of money, an identified quantity, etc.)?
3. How can I best represent the important problem information? (Some examples below.)
   a. Do I want to show something over time?
   b. Do I want to show proportional amounts?
   c. Do I want to show a pattern?
4. Do I want the visual to lead to mathematical notation or should it lead to the answer without mathematical notation?
5. Choose a visual keeping in mind the following:
   a. Does it organize the information?
   b. Does it display the important information so the learner can solve for the unknown?
   c. Does it simplify the problem structure for the learner?

Reflection question:

1. Did my selected visual address the following:
   a. Organize the information in the problem?
   b. Display the important information in the problem so that the learner can solve for the unknown?
   c. Simplify the problem structure for the learner?

Handout 1B.1
Money Problem

Andi spent $\frac{3}{8}$ of her money on a baseball glove. Then she spent $8 of what was remaining on a movie ticket. Now she has $32. How much money did she start with?

Answer these questions:

Question	Answer	
1. What is the problem about (e.g., rate, proportion/ ratio, money spent, etc.)?		
2. What is unknown (e.g., starting amount of money? An identified quantity, etc.)?		
3. How can I best represent the important problem information?	(circle Yes or No)	
Do I want to show something over time?	Yes	No
Do I want to show proportional amounts?	Yes	No
Do I want to show a pattern?	Yes	No
4. Do I want the visual representation to lead to mathematical notation or should it lead to the answer without mathematical notation?		
5. Choose a visual representation keeping in mind the following: • Does it organize the information? • Does it display the important information, so the learner can solve for the unknown? • Does it simplify the problem structure for the learner?		
Show visual here.		

Handout 1B.2a
Prepare to Share (Option 1: Recommended Activity)

Directions: Use the questions below to select a visual representation for a mathematics problem from your curriculum. Answer the questions by writing an answer in the right-hand column. Then create your visual representation based on answers to your questions. For the purpose of sharing during the next session's **Debrief** segment, please answer these questions.

Problem and Description of the context: _____

Answer these questions:

Question	Answer
1. What is the problem about (e.g., rate, proportion/ ratio, money spent, etc.)?	
2. What is unknown (e.g., starting amount of money? An identified quantity, etc.)?	
3. How can I best represent the important problem information?	
4. Do I want the visual to lead to mathematical notation or should it lead to the answer without mathematical notation?	
5. Choose a visual representation keeping in mind the following: • Does it organize the information? • Does it display the important information, so the learner can solve for the unknown? • Does it simplify the problem structure for the learner?	
Show your visual representation:	

Handout 1B.2a (Continued)

1. Tell how your visual representation helped to organize the information in the problem.

2. Tell how your visual representation displays the important information in the problem so that the learner can solve for the unknown.

3. Tell how the visual representation helped to simplify the problem structure for the learner.

Handout 1B.2b
Prepare to Share (Option 2: Custom Activity)

Directions: You may opt to develop a different activity to reinforce the content of this session. If so, consider the following questions to keep the activity on target:

- How does this activity relate to Recommendation 3, *How-to* Steps 1 and 3?
- What products will participants collect or develop and bring back to discuss during the next session?

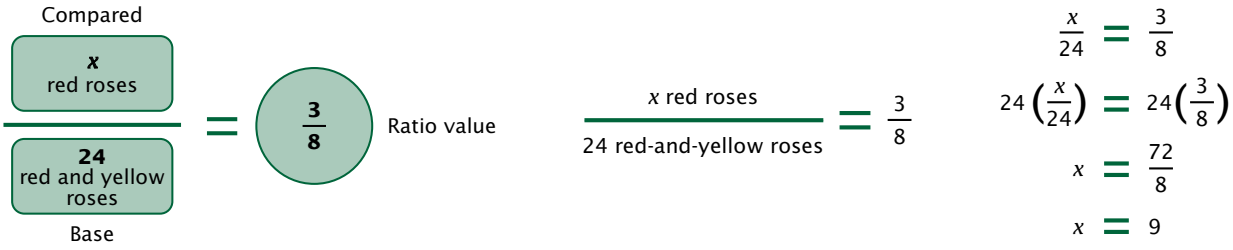
After implementing the activity, complete the items below to refer to during the **Debrief** segment of the next PLC session.

1. Explain your experience completing the activity. Be sure to address issues such as the level of difficulty of the activity, any problems you encountered, etc.

2. Discuss what you learned from completing this activity.

Handout 2A.1
Visual Representation Question Worksheet

Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?



Answer these questions:

Question	Answer
1. What is the problem about (e.g., rate, proportion/ ratio, money spent, etc.)?	
2. What is unknown (e.g., start amount of money? identified quantity, etc.)?	
3. Does the visual representation address the important problem information? • Did it need to show something over time? • Did it show proportional amounts? • Did it need to show a pattern?	
4. Did the visual representation lead to mathematical notation?	
5. Visual representations should be selected with the following 3 reflection questions in mind. Did this visual accomplish this? • Did it organize the information? • Did it display the important information, so the learner can solve for the unknown? • Did it simplify the problem structure for the learner?	
6. Is there an alternative visual representation you might have chosen? If so, show it here.	

Handout 2A.2
Visual Representation Think-Aloud Analysis

Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

Answer these questions (refer to the think-aloud in the practice guide on pp. 27–28):

Question	Yes/No
1. Did the teacher in her think-aloud tell what the problem is about (e.g., rate, proportion/ratio, money spent, etc.)?	
2. Did the teacher in her think-aloud tell how she figured out what the problem type is about?	
3. Did the teacher in her think-aloud explain how she determined the unknown?	
4. Did she reread parts of the problem to confirm the unknown and/or the problem type?	
5. Did the teacher explain why she selected the visual representation? If so, what did she say?	
6. Did she tell how the visual representation simplified, organized, and displayed the important information?	
7. Did the teacher in her think-aloud sum up and review her problem solving?	
8. Was there anything critical you think she forgot to say in her think-aloud? If so, list below:	

Handout 2A.3

Video Viewing Guide – Visual Representation Think-Alouds

Grade level viewing (circle one):

Seventh-Grade Class

Problem presented: Sally likes to exercise. For every 10 minutes that she runs, she does jumping jacks for 2 minutes. If she exercises for 1 hour, how many minutes did she do jumping jacks?

Eighth-Grade Class

Problem presented: Andi spent $\frac{3}{8}$ of her money on a baseball glove. Then she spent $\frac{1}{5}$ of what was remaining on a movie ticket. Now she has $32. How much money did she start with?

Directions: The teacher is using a think-aloud to demonstrate how she selected a visual representation to help her solve a problem. As you view the lesson, think about the questions below. In each box, record how the teacher addresses these questions. After the video, take one minute to record any thoughts about the video.

How did the teacher explain what the problem was about and how she knew that?	How was the unknown identified and how did the teacher explain how she figured it out?

Handout 2A.3 (Continued)

What visual representation did the teacher select? How did the teacher justify why she selected that visual representation?	How did the teacher explain how the visual representation simplified, organized, or helped her to display important information and how she knew that?
What equation did the teacher write to accompany the visual representation (if any)?	How did the teacher sum up and review her problem solving to confirm her solution method?

Record: Participants will record their thoughts below after viewing the video.

Handout 2A.4
Table Representation for Ratio Problem

Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

Monica's table:

Number of red roses	Number of yellow roses	Total number of roses
3	5	8
6	10	16
9	15	24

Answer: She had 9 red roses.

Answer these questions:

Question	Answer
1. What is the problem about (e.g., rate, proportion/ ratio, money spent, etc.)?	*ratio*
2. What is unknown (e.g., start amount of money? An identified quantity, etc.)?	*Red roses in Monica's bouquet*
3. Does the visual representation address the important information in the problem? • Did it need to show something over time? • Did it show proportional amounts? • Did it need to show a pattern?	
4. Does the visual representation lead to mathematical notation?	
5. Did the visual representation address the following? • Did it organize the information? • Did it display the important information, so the learner can solve for the unknown? • Did it simplify the problem structure for the learner?	
6. Is there an alternative visual representation you might choose for Monica and Bianca's problem? If so, show it here.	

Handout 2B.1
Preparing a Think-Aloud

Directions: Refer to your answers on **Handout 2A.4: Table Representation for Ratio Problem** to write a think-aloud. Use the prompts in the left-hand column to write notes in the right-hand column for content you **plan** to include in the think-aloud. Use the notes in the right-hand column to write your think-aloud on page 17. The think-aloud should model your thinking for how you would select the table as a visual representation to solve the problem.

Prompts	Think-Aloud Ideas
Explain what the problem is about and how you knew what the problem is about.	
Identify the unknown and explain how you identified it.	

Handout 2B.1 (Continued)

Prompts	Think-Aloud Ideas
Explain why you are selecting the visual representation (table); include how it simplifies, organizes, and displays important problem information.	
Connect the visual representation to mathematical notation.	
Summarize and review the problem solving to confirm the solution.	

Handout 2B.1 (Continued)

1. Write a think-aloud using your notes above:

Handout 2B.2

Table Visual Representation for Ratio Problem with Altered Numbers

Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with ten dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

Monica's table:

Number of red roses	Number of yellow roses	Total number of roses
3	5	8
6	10	16
9	15	24
12	20	32
15	25	40
18	30	48
21	35	56
24	40	64
27	45	72
30	50	80
33	55	88
36	60	96
39	65	104
42	70	112
45	75	120

Handout 2B.3

Preparing a Think-Aloud for Instruction (Option 1, Recommended Activity)

Directions: To reinforce the key concepts presented in this session, you will implement the following activity in your classroom before the next PLC session:

- Choose a mathematics problem.
- Choose a problem that can be taught using visual representations to solve it.
- Using the table below, develop a think-aloud that you will use to model how to select the visual representation to help your students with solving the problem.

In the **Debrief** segment of the next PLC session, be prepared to share your work.

Preparing the Think-Aloud:

1. Write the problem you selected.

2. Draw the visual representation you have selected.

3. Use the guiding questions in the left-hand column on the table below to write a think-aloud. Write bullet points of the important information you wish to cover in the think-aloud in the right-hand column. You will use this think-aloud as you present the visual representation to your students.

Prompts	Think-Aloud Ideas
Explain what the problem is about and how you knew what the problem is about.	

Handout 2B.3 (Continued)

Prompts	Think-Aloud Ideas
Identify the unknown and explain how you identified it.	
Explain why you are selecting the visual representation (table); include how it simplifies, organizes, and displays important problem information.	

Handout 2B.3 (Continued)

Prompts	Think-Aloud Ideas
Connect the visual representation to mathematical notation.	
Summarize and review the problem solving to confirm the solution.	

4. Write your think-aloud using your notes above:

Handout 2B.4a
Prepare to Share (Option 1, Recommended Activity)

Prepare to share the following:

1. What led you to selecting the visual representation and did you consider any alternate visual representations for this problem?

2. Describe the lesson you taught. (Use Handout 2B.3 to guide you.)

3. Did you teach the lesson as **planned**? If not, describe any adjustments you made and why you made them.

4. Did your students practice solving similar problems with the visual representation you presented?

Handout 2B.4b
Prepare to Share (Option 2, Custom Activity)

Directions: You may opt to develop a different activity to reinforce the content of this session. If so, consider the following questions to keep the activity on target:

- How does this activity relate to Recommendation 3, *How-to* Step 2?
- What products will participants collect or develop and bring back to discuss during the next session?

After implementing the activity, complete the items below to refer to during the **Debrief** segment of the next PLC session.

1. Explain your experience completing the activity. Be sure to address issues such as the level of difficulty of the activity, any problems you encountered, etc.

2. Discuss what you learned from completing this activity.

Handout 3A.1
Video Viewing Guide – Introduction to Recommendation 4

Directions: In this video, John Woodward, a panel member for the Improving Problem-Solving practice guide, provides a brief overview of Recommendation 4, including a description of the three *How-to* steps for carrying out the recommendation. As you watch the video, take notes on the important information about each *How-to* step.

Recommendation 4: Expose students to multiple problem-solving strategies.

Topic Area	Notes
***How-to* Step 1:** Provide instruction in multiple strategies.	
***How-to* Step 2:** Provide opportunities for students to compare multiple strategies in worked examples.	

Handout 3A.1 (Continued)

Topic Area	Notes
***How-to* Step 3:** Ask students to generate and share multiple strategies for solving a problem.	
Roadblocks or other things of note:	

Handout 3A.2
Choosing Worked Examples and Understanding the Benefits

Chosing Apropriate Worked Examples and Understanding the Benefits

What is the focus?	**1** To ***introduce/teach*** a new mathematical concept or skill using a new solution method.	**2** To ***review*** two or more solution methods.
Choose and approach:	**A** Compare a visual representation to an algorithm.	**B** Compare two algorithms that highlight different solution paths.

What are the benefits of the approach for each focus?

1 A	1 B	2A	2B
Help students make sense of the algorithm	Introduce new mathematical solutions to show more than one correct strategy	Reinforce the connection between procedural and conceptual representations	Support the understanding that problems have more than one solution
Connect the concepts underlying visual representations to algorithms	Make connections across correct mathematical solutions to build mathematical understanding		Build efficiency for selecting appropriate solution strategies
		Comparison activities can be used for efficient review before increasing problem or skill difficulty	

Overall benefits:

- ✓ Expose students to multiple, correct solution strategies without overtaxing students' cognition
- ✓ Promote positive discussion about mathematics and problem solving
- ✓ Enhance analytical skills
- ✓ Build flexibility for problem solving

Handout 3A.3

Guiding Questions for Designing Worked Example Comparisons

1. **When selecting the mathematical skill or topic**
   - **a.** Am I introducing something new or reviewing?
   - **b.** What mathematical concept(s) or skills do I want to highlight?
   - **c.** What background knowledge do my students already have (i.e., what do my students already understand; what do they not understand)?
2. **When selecting the solutions for the topic**
   - **a.** Could I include at least one solution that includes a visual representation? Why or why not?
   - **b.** What features of each solution might I focus on to highlight the mathematical concepts included in each solution
   - **c.** In what ways is the comparison of these two solutions important for students' progress in problem solving and mathematical development?

Handout 3A.4
Examining Example 15

See below for Example 15: The teacher presents two worked examples to highlight the concept of composite variables. She labels the two solutions, Mandy's solution and Erica's solution (p. 35 of practice guide).

Mandy's solution		Erica's solution	
$5(y + 1) = 3(y + 1) + 8$		$5(y + 1) = 3(y + 1) + 8$	
$5y + 5 = 3y + 3 + 8$	Distribute	$2(y + 1) = 8$	Subtract on both
$5y + 5 = 3y + 11$	Combine	$y + 1 = 4$	Divide on both
$2y + 5 = 11$	Subtract on both	$y = 3$	Subtract on both
$2y = 6$	Subtract on both		
$y = 3$	Divide on both		
Teacher: Mandy and Erica solved the problem differently, but they got the same answer. Why? Would you chose to use Many's way or Erica's way? Why?			

Complete the Table Below: Apply the 6 guiding questions for designing worked example comparisons (left hand column) to the worked example comparison by evaluating the teacher's selection. When completing the table, think about the two solutions the teacher chose, and the answers to questions that are provided. Write a "participant answer" for questions that are left blank.

When selecting the mathematical skill or topic	Participant Answer
Is the teacher introducing something new or reviewing material?	*Introducing something new*

Handout 3A.4 (Continued)

When selecting the mathematical skill or topic	Participant Answer
What mathematical concept(s) is the teacher highlighting?	*Composite variables or composite quantity*
What background knowledge do you think these students already have (i.e., what do they already understand)?	
Did she choose solutions that include: Two algorithms? A visual and an Algorithm? Why do you think she made that choice?	*Two algorithms*

Handout 3A.4 (Continued)

When selecting the mathematical skill or topic	Participant Answer
What features of each solution should the teacher focus on when comparing these two solutions?	
How does comparing these two solutions support students' progress in problem solving and mathematical development?	

Handout 3A.5
Designing a Worked Example for Fraction Division

Situation: Your class has been working on fraction division. You've taught dividing a whole number by a fraction (whole number as dividend and fraction as a divisor). You are about to introduce dividing a fraction by a fraction but would like to review the concept of fraction division using the problem $5 \div \frac{1}{2}$. You are designing a worked example comparison to show solutions for solving $5 \div \frac{1}{2}$ to highlight the concept of fraction division and to tie it to an algorithm.

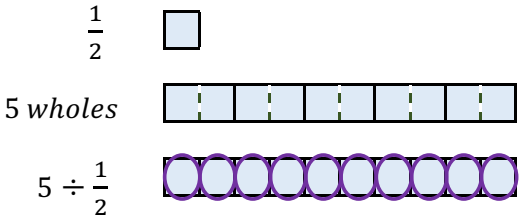
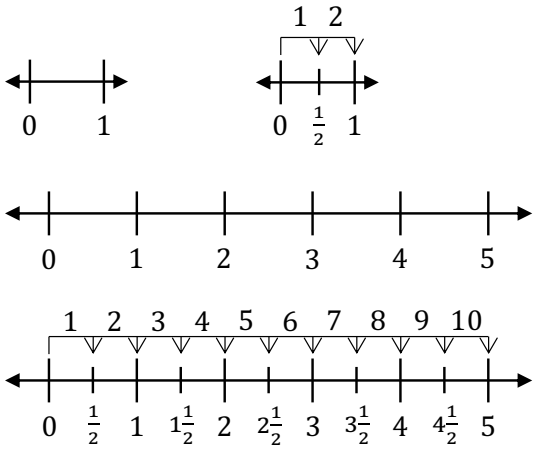
Directions: Given the situation above, please select two solutions from the options below that you would use in a side-by-side comparison activity with your students (circle your choices). You can select one algorithm and one visual, or you may select two algorithms (You do not have to choose two in the same category). Refer to the diagram on **Handout 3A.2: Choosing Worked Examples and Understanding the Benefits** and consider the benefits in column 2A vs column 2B. Complete the table starting on page 34 as you consider your selection of solutions.

Algorithm Solutions

Reciprocal Method	Common Denominator Method
$5 \div \frac{1}{2}$ $= \frac{5}{1} \div \frac{1}{2}$ $= \frac{5}{1} \times \frac{2}{1}$ $= \frac{10}{1}$ $= 10$	$5 \div \frac{1}{2}$ $= \frac{5}{1} \div \frac{1}{2}$ $= \frac{10 \div 1}{2 \div 2}$ $= \frac{10}{1}$ $= 10$

Handout 3A.5 (Continued)

Visual Representation Solutions

Diagram Method (Area Model)	Number Line Method
 There are 10 groups of $\frac{1}{2}$ in 5 wholes. $5 \div \frac{1}{2} = 10$	 There are 10 groups of $\frac{1}{2}$ in 5 wholes. $5 \div \frac{1}{2} = 10$

Handout 3A.5 (Continued)

When selecting the mathematical skill or topic	Participant Answer
Am I introducing something new or reviewing?	*Reviewing fraction division dividing a whole number by a fraction*
What mathematical concept(s) do I want to highlight?	*Concept of division with fractions*
What background knowledge do my students already have (i.e. what do my students already understand; what do they not understand)?	
Should I compare: Two algorithms? A visual and an Algorithm? Why?	

Handout 3A.5 (Continued)

When selecting the mathematical skill or topic	Participant Answer
What features of each solution should I focus on when comparing the two solutions?	
How does comparing these two solutions support students' progress in problem solving and mathematical development?	

Handout 3B.1a
Designing A Worked Example Comparison (Concept A)

Math concept: fraction division

Solve: $1\frac{1}{2} \div \frac{1}{8}$

Solution 1	Solution 2

When selecting the mathematical skill or topic	Participant Answer
Am I introducing something new or reviewing?	
What mathematical concept(s) do I want to highlight?	

Handout 3B.1a (Continued)

When selecting the mathematical skill or topic	Participant Answer
What background knowledge do my students already have (i.e., what do my students already understand)?	
Should I compare: Two algorithms? A visual and an algorithm? Why?	
What features of each solution should I focus on when comparing the two solutions?	
How does comparing these two solutions support students' progress in problem solving and mathematical development?	

Handout 3B.1b
Designing A Worked Example Comparison

Math concept: Comparing ratios

The Buy-It-Here Store sells fruit strips. They come in purple, red, and green strips.

Fruit Strips
Purple Strips 10 for $4
Red Strips 18 for $8
Green Strips 28 for $12

Which one is the best deal?

Solution 1	Solution 2

Handout 3B.1b (Continued)

When selecting the mathematical skill or topic	Participant Answer
Am I introducing something new or reviewing?	
What mathematical concept(s) do I want to highlight?	
What background knowledge do my students already have (i.e., what do my students already understand)?	

Handout 3B.1b (Continued)

When selecting the solutions for the topic	Participant Answer
Should I compare: Two algorithms? A visual and an algorithm? Why?	
What features of each solution should I focus on when comparing the two solutions?	
How does comparing these two solutions support students' progress in problem solving and mathematical development?	

Handout 3B.2
Solution Bank

Concept A:

Robert's Solution
(Reciprocal Method)

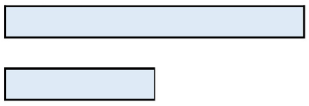
$$1\frac{1}{2} \div \frac{1}{8}$$
$$= \frac{3}{2} \div \frac{1}{8}$$
$$= \frac{3}{2} \times \frac{8}{1}$$
$$= \frac{24}{2}$$
$$= 12$$

Angela's Solution
(Common Denominator Method)

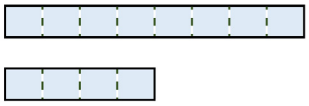
$$1\frac{1}{2} \div \frac{1}{8}$$
$$= \frac{3}{2} \div \frac{1}{8}$$
$$= \frac{12}{8} \div \frac{1}{8}$$
$$= \frac{12 \div 1}{8 \div 8}$$
$$= \frac{12}{1}$$
$$= 12$$

Maria's Solution
(Diagram Method)

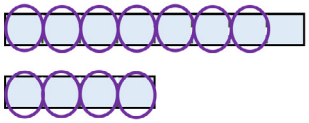
$1\frac{1}{2}$



$1\frac{1}{2} \div \frac{1}{8}$



$1\frac{1}{2} \div \frac{1}{8}$



There are 12 groups of $\frac{1}{8}$ in $1\frac{1}{2}$.

$1\frac{1}{2} \div \frac{1}{8} = 12$

Handout 3B.2 (Continued)

Concept B:

Jasmine's Solution

Unit price per strip:

Purple $\frac{\$4.00}{10\ strips} = \$0.40\ each$

Red $\frac{\$8.00}{18\ strips} = \$0.44\ each$

Green $\frac{\$12.00}{28\ strips} = \$0.43\ each$

→ Purple fruit strips are the best deal.

Anthony's Solution

Purple strips:

10	10	10	10	10
$4	$4	$4	$4	$4

versus Red strips:

18	18	18	18	18	18	18
$8	$8	$8	$8	$8	$8	$8

→ $8 worth of Purple is 20 strips, and $8 worth of Red is 18 strips.

Purple strips:

10	10	10	10	10
$4	$4	$4	$4	$4

versus Green strips:

28	28	28	28	28	28	28
$12	$12	$12	$12	$12	$12	$12

→ $12 worth of Purple is 30 strips, and $12 worth of Green is 28 strips.

Angela's Solution

Purple

Cost ($)	# of strips
4	10
8	20
12	30
16	40
20	50
24	60

Red

Cost ($)	# of strips
8	18
16	38
24	54
32	72
40	90
48	108

Green

Cost ($)	# of strips
12	28
24	56
36	84
48	112
60	140
72	168

→ $24 worth of Purple is 60 strips, $24 worth of Red is 54 strips, and $24 worth of Green is 56 strips.

→ Purple fruit strips are the best deal.

Reece's Solution

Purple $\frac{10\ strips}{\$4.00}$

Red $\frac{18\ strips}{\$8.00}$

Green $\frac{28\ strips}{\$12.00}$

→ The least common denominator for $4, $8, and $12 is $24.

Purple $\frac{10\ strips}{\$4.00} = \frac{60\ strips}{\$24.00}$

Red $\frac{18\ strips}{\$8.00} = \frac{54\ strips}{\$24.00}$

Green $\frac{28\ strips}{\$12.00} = \frac{56\ strips}{\$24.00}$

$\frac{60\ purple\ strips}{\$24.00} > \frac{56\ green\ strips}{\$24.00} > \frac{54\ red\ strips}{\$24.00}$

→ Purple fruit strips are the best deal.

Handout 3B.3a
Prepare to Share (Option 1, Recommended Activity)

Math concept: Math problem

Solution 1	Solution 2

When selecting the mathematical skill or topic	Participant Answer
Am I introducing something new or reviewing?	
What mathematical concept(s) do I want to highlight?	
What background knowledge do my students already have (i.e., what do my students already understand)?	

Handout 3B.3a (Continued)

When selecting the solutions for the topic	Participant Answer
Should I compare: Two algorithms? A visual and an algorithm? Why?	
What features of each solution should I focus on when comparing the two solutions?	
How does comparing these two solutions support students' progress in problem solving and mathematical development?	

Handout 3B.3a (Continued)

Prepare to Share:

1. Was selecting a problem to highlight this concept easy for you? Challenging for you? What are the issues?

2. What questions and considerations did you find most useful when designing the activity?

3. What do you expect might be the response you will get from students when presenting this activity?

Handout 3B.3b
Prepare to Share (Option 2, Recommended Activity)

Directions: You may opt to develop a different activity to reinforce the content of this session. If so, consider the following questions to keep the activity on target:

- How does this activity relate to Recommendation 4, *How-to* Steps 1 and 2?
- What products will participants collect or develop and bring back to discuss during the next session?

After implementing the activity, complete the items below to refer to during the **Debrief** segment of the next PLC session.

1. Explain your experience completing the activity. Be sure to address issues such as the level of difficulty of the activity, any problems you encountered, etc.

2. Discuss what you learned from completing this activity.

Handout 4A.1
Comparing Solutions Framework and Graphic Organizer

In the table below, answer the questions for each solution.

Features of the Solution Strategies	Solution 1	Solution 2
How was the problem solved? (e.g., visual representation, algebraic equation, table, etc.)		
Which operation(s) were used in each solution?		
Did one of the solution methods rely on a pattern?		

In the table below, write ways in which the two solution methods are similar and different.

Similar	Different

Handout 4A.2

The Rate Problem and Anticipatory Questions – Part A

In one day, you earn \$76 for 8 hours of work. You work 37.5 hours for the entire week. What is your weekly pay?

Method 1	Method 2
Unit rate: $\$76 \div 8\ hours = \$9.50\ per\ hour$ $37.5\ hours \times \$9.50\ per\ hour = \$356.25$	See table below

Hours	Pay	
8	\$76	
16	\$152	\$76
24	\$228	\$76
32	\$304	\$76
36	\$342	\$38
37	\$351.50	\$9.50
37.5	\$356.25	\$4.75

Handout 4A.2 (Continued)
The Rate Problem and Anticipatory Questions – Part B

Anticipatory Questions:

1.

2.

3.

Handout 4A.3

Video Viewing Guide – Teaching Students to Compare Solution Methods for the Rate Problem

Directions: Record up to 5 questions the teacher asks the student that you found to be helpful and supportive of the student's comparison. Record these in the Question column. In the Reaction column, write how the question helped the student identify what is similar or different about the solution methods or whether you felt the question was effective in helping the teacher meet the mathematical goals of the lesson. Did you anticipate any questions that the teacher posed?

Question	Reaction
1.	
2.	

Handout 4A.3 (Continued)

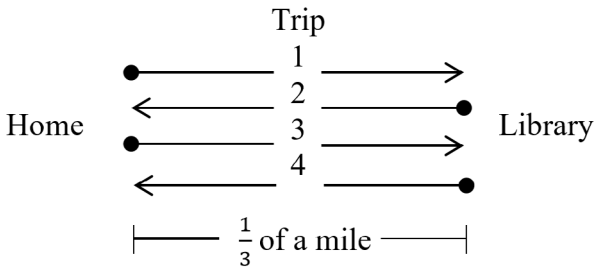
Question	Reaction
3.	
4.	
5.	

Handout 4B.1

Video Viewing Guide – Comparing the Fractions Problem

Jenny lives $\frac{1}{3}$ of a mile away from the library. On Saturday, she made two trips to the library and then home. Her first trip was to check out a book and her second trip was to return the book. How far did Jenny walk after taking these two trips to the library?

First trip	Home to library	$\frac{1}{3}$ of a mile
	Library to home	$\frac{1}{3}$ of a mile
Second trip	Home to library	$\frac{1}{3}$ of a mile
	Library to home	$\frac{1}{3}$ of a mile



$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$

$1\frac{1}{3}$ miles

$4 \times \frac{1}{3} = \frac{4}{3}$

$1\frac{1}{3}$ miles

Reflection:

1. What does the teacher do to encourage the student's skill in comparing?

2. What does the teacher do to advance and promote students' mathematical thinking?

Handout 4B.2
Fruit Strips Problem

Directions: PLC participants will apply the framework and graphic organizer to the Fruit Strips Problem and will also write probing questions that address critical mathematics learning.

Problem: The Buy-It-Here Store sells fruit strips. They come in purple, red, and green strips. Which one is the best deal?

Fruit Strips

Purple Strips 10 for $4

Red Strips 18 for $8

Green Strips 28 for $12

Angela's Solution

Purple

Cost ($)	# of strips
4	10
8	20
12	30
16	40
20	50
24	60

Red

Cost ($)	# of strips
8	18
16	38
24	54
32	72
40	90
48	108

Green

Cost ($)	# of strips
12	28
24	56
36	84
48	112
60	140
72	168

→ $24 worth of Purple is 60 strips, $24 worth of Red is 54 strips, and $24 worth of Green is 56 strips.

→ Purple fruit strips are the best deal.

Reece's Solution

Purple $\frac{10\ strips}{\$4.00}$

Red $\frac{18\ strips}{\$8.00}$

Green $\frac{28\ strips}{\$12.00}$

→ The least common denominator for $4, $8, and $12 is $24.

Purple $\frac{10\ strips}{\$4.00} = \frac{60\ strips}{\$24.00}$

Red $\frac{18\ strips}{\$8.00} = \frac{54\ strips}{\$24.00}$

Green $\frac{28\ strips}{\$12.00} = \frac{56\ strips}{\$24.00}$

$\frac{60\ purple\ strips}{\$24.00} > \frac{56\ green\ strips}{\$24.00} > \frac{54\ red\ strips}{\$24.00}$

→ Purple fruit strips are the best deal.

Handout 4B.3

Fruit Strips Problem – Comparing Solutions Framework and Graphic Organizer

Features of the Solution Strategies	Solution 1	Solution 2
How was the problem solved? (e.g., visual representation, algebraic equation, table, etc.)		
Which operation(s) were used in each solution?		
Did one of the solution methods rely on a pattern?		

Use the table above to write some ways the two solution methods are similar and/or different. Remember to focus on how the mathematics skills and concepts underlying each strategy are similar or different.

Similar	Different

Handout 4B.3 (Continued)

Question	Reaction
1. 24 is a common multiple for 4, 8, and 12 and was used in both solution methods. **2.** Common multiples are used to find common denominators; 24 is a critical number in each solution method. **3.** Reece used equivalent ratios and Angela used a common multiple approach. **4.** Angela used a table to organize her information. **5.** Both solutions solve each fruit strip color separately to find how many can be bought for the same amount of money ($24).	**1.** Do you see any numbers that are the same in both solution methods that were not given in the problem?

Handout 4B.4
Problems and Solutions

Sixth-Grade Problem:

Ms. Freeman's math class has a male student to female student ratio of 3:2. If her class has 12 boys, how many girls does she have?

<table>
<tr><th>Angela's Solution</th><th>Angela's Solution</th><th>Reece's Solution</th></tr>
<tr>
<td>Let x = number of girls

$\frac{3}{2} = \frac{12}{x}$

$\frac{(3 \times 4)}{(2 \times 4)} = \frac{12}{x}$

$x = 8$
Her class has 8 girls.</td>
<td>
<table>
<tr><th>Number of boys</th><th>Number of girls</th></tr>
<tr><td>3</td><td>2</td></tr>
<tr><td>6</td><td>4</td></tr>
<tr><td>9</td><td>6</td></tr>
<tr><td>12</td><td>8</td></tr>
</table>
</td>
<td>Let x = number of girls

$\frac{3}{2} = \frac{12}{x}$

$3x = 24$

$x = 8$
Her class has 8 girls.</td>
</tr>
</table>

Handout 4B.4 (Continued)

Seventh-Grade Problem:

In a bag of trail mix, the ratio of banana chips to dried apple slices is 4 to 12. In another bag of trail mix, the ratio of banana chips to dried apple slices is 2 to 5. Both bags have the same number of pieces. Which bag has more dried apple slices?

<table>
<tr><th>Rebeca's Solution</th><th>Nancy's Solution</th></tr>
<tr><td>Each bag has 2 parts: banana chips and dried apple slices.

4 + 12 = 16 total parts of the whole.
12 ÷ 16 = 0.75 = 75% of bag 1 is apple slices.

2 + 5 = 7 total parts of the whole.
5 ÷ 7 = 0.714 = 71.4% of bag 2 is apple slices.

The percent of dried apple slices is greater in bag 1. Bag 1 has more dried apple slices.</td><td>Bag 1
<table><tr><th># of banana chips</th><th># of dried apple slices</th></tr><tr><td>4</td><td>12</td></tr></table>
Bag 2
<table><tr><th># of banana chips</th><th># of dried apple slices</th></tr><tr><td>2</td><td>5</td></tr><tr><td>4</td><td>10</td></tr></table>
By doubling the amount of banana chips in bag 2, the number of dried apple slices is greater than the number of dried apple slices in bag 1.</td></tr>
<tr><th colspan="2">Tyler's Solution</th></tr>
<tr><td colspan="2">Bag 1:
$\frac{\text{(number of dried apple slices)}}{\text{(number of banana chips)}} = \frac{12}{4} = \frac{3}{1}$

The unit rate for bag 1 is 3 dried apple slices for every banana chip.

Bag 2:
$\frac{\text{(number of dried apple slices)}}{\text{(number of banana chips)}} = \frac{5}{2} = \frac{2.5}{1}$

The unit rate for bag 2 is 2.5 dried apple slices for every banana chip.

The unit rate for bag 1 is greater than the unit rate for bag 2. Thus, bag 1 will have more dried apple slices.</td></tr>
</table>

Handout 4B.5

Designing a Worked Example Lesson Plan (Option 1: Recommended Activity)

Problem:

Solution 1	Solution 2

Handout 4B.6
Comparing Solutions Framework and Graphic Organizer

Features of the Solution Strategies	Solution 1	Solution 2
How was the problem solved? (e.g., visual representation, algebraic equation, table, etc.)		
Which operation(s) were used in each solution?		
Did one of the solution methods rely on a pattern?		

Use the table above to write some ways the two solution methods are similar and/or different. Remember to focus on how the mathematics skills and concepts underlying each strategy are similar or different.

Similar	Different

Handout 4B.6 (Continued)

What do I want the students to learn or notice when comparing these solutions?	What probing questions could be posed to elicit this learning?

Handout 4B.7a
Prepare to Share (Option 1: Recommended Activity)

Prepare to Share the Following:

1. What led you to select this problem and the two solutions?

2. Describe the lesson you taught. (Use **Handout 4B.6: Comparing Solutions Framework and Graphic Organizer** to guide you.)

3. Did you teach the lesson as **planned** (e.g., did you use the probing questions you anticipated)? If not, describe any adjustments you made and why you made them.

Handout 4B.7b
Prepare to Share (Option 2: Custom Activity)

Directions: You may opt to develop a different activity to reinforce the content of this session. If so, consider the following questions to keep the activity on target:

- How does this activity relate to Recommendation 4, *How-to* Step 2?
- What products will participants collect or develop and bring back to discuss during the next session?

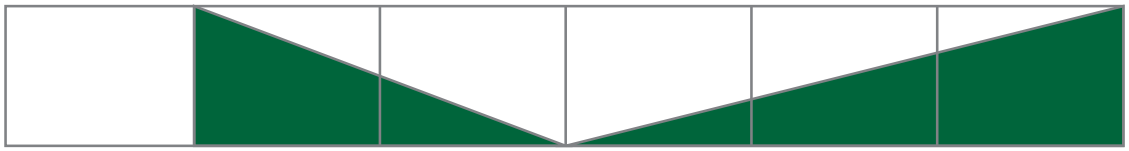
After implementing the activity, complete the items below to refer to during the **Debrief** segment of the next PLC session.

1. Explain your experience completing the activity the PLC developed to reinforce the content of the session. Be sure to address issues such as the level of difficulty of the activity and any problems you encountered.

2. Discuss what you learned from completing this activity.

Handout 5A.1
Examining Example 17 – Fraction Strip

What fraction of the whole rectangle is green?



Student 1
$(\frac{1}{3} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) = \frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$ of the entire rectangle.
Student 2
$\frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$

Part 1: Can you draw any conclusions about the students' thinking and understanding just from the equations?

Part 2: Now look at the two explanations the students provided (Example 17, p. 37). What conclusions can you draw about the strategy the student used to solve the problem?

Student 1's explanation:

Student 2's explanation:

Part 3: Now look at the suggested questions in the practice guide on page 37 and answer the questions below.

For **Student 1**'s response, the practice guide recommends asking the student how they knew $\frac{1}{3}$ was the green part on the left. It also suggests asking, "How did you know the green part on the right is half the area?"

For **Student 2'**s response, the practice guide recommends asking the student, "How did you know that the green part is the same as the area colored black?"

What insight about each student's strategies is the teacher trying to uncover by posing these additional questions to her students?

Handout 5A.1 (Continued)

Part 4: What additional questions might you pose to each of these students to elicit more information about the strategy they used?

Student 1's explanation:

Student 2's explanation:

Handout 5A.2
Questioning Framework for Mathematics

Question Type	Description
Gathering Information	*Students recall facts, definitions or procedure.*
Probing Thinking	*Students explain, elaborate or clarify their thinking, including articulating the steps in a solution method or the completion of a task.*
Making the Mathematics Visible	*Students discuss the mathematical structures and make connections among mathematical ideas and relationships.*
Encouraging Reflection and Justification	*Students reveal deeper understanding of their reasoning and actions, including making an argument for the validity of their work.*

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

Types of Probing Questions During Problem Solving

Probing Question Type	Description	Example
Clarifying	*Encourage students to clarify a statement or approach.*	*"How?" or "What do you mean?" or "So?…tell me more."*
Confirmation	*Repeating something the student said in a new way to confirm the teacher understood the student's approach.*	
Leading	*Can get students to think about what their next step might be when solving a problem or what conclusions they could draw from what they just did and why it did or did not work. (This is most appropriate if questions are being used to help a student complete a challenging problem.)*	
Elaboration or Extended Feedback	*When a teacher paraphrases and summarizes students' thinking it is a form of elaborating on what the student said and extending their answer through feedback.*	

Handout 5A.3
Probing Questions for Example 9

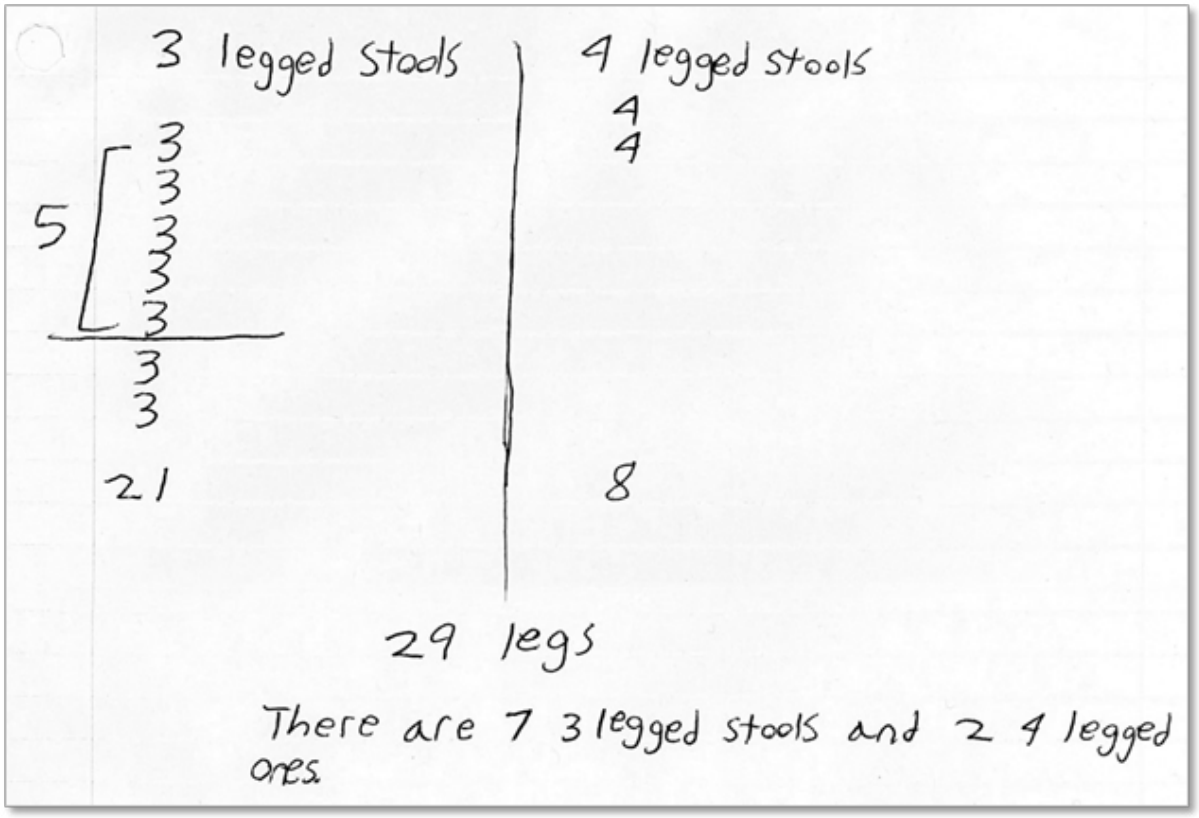
Clarifying questions:	**Leading questions:**
Confirmation questions:	**Elaboration or Extended Feedback:**

Handout 5B.2
Video Viewing Guide – Proportion Problem I

Directions: Take a few minutes to review the solution to Proportion Problem I.

> Ramona's furniture store has a choice of 3-legged stools and 4-legged stools. There are five more 3-legged stools than 4-legged stools. When you count the legs of the stools, there are exactly 29 legs. How many 3-legged and 4-legged stools are there in the store?

Student Work:



Handout 5B.2

Video Viewing Guide – Proportion Problem I – Focus on Student Explanation

Directions: As you view the video, determine whether the student exhibited any of the behaviors listed in Column 3 (**Did the student…**) while explaining her solution. Check the appropriate column to indicate whether the student's statement was made with or without teacher support. Leave the item blank, if the student did not exhibit the behavior.

Without Support	With Teacher Support	Did the student…
		Did the student state what the problem was about?
		Did the student state what she is trying to find?
		Did the student talk about which information is important for solving the problem?
		Did the student explain her strategy as it relates to key information in the problem?
		Did the student explain why she thinks her answer is reasonable?

Notes about the interaction:

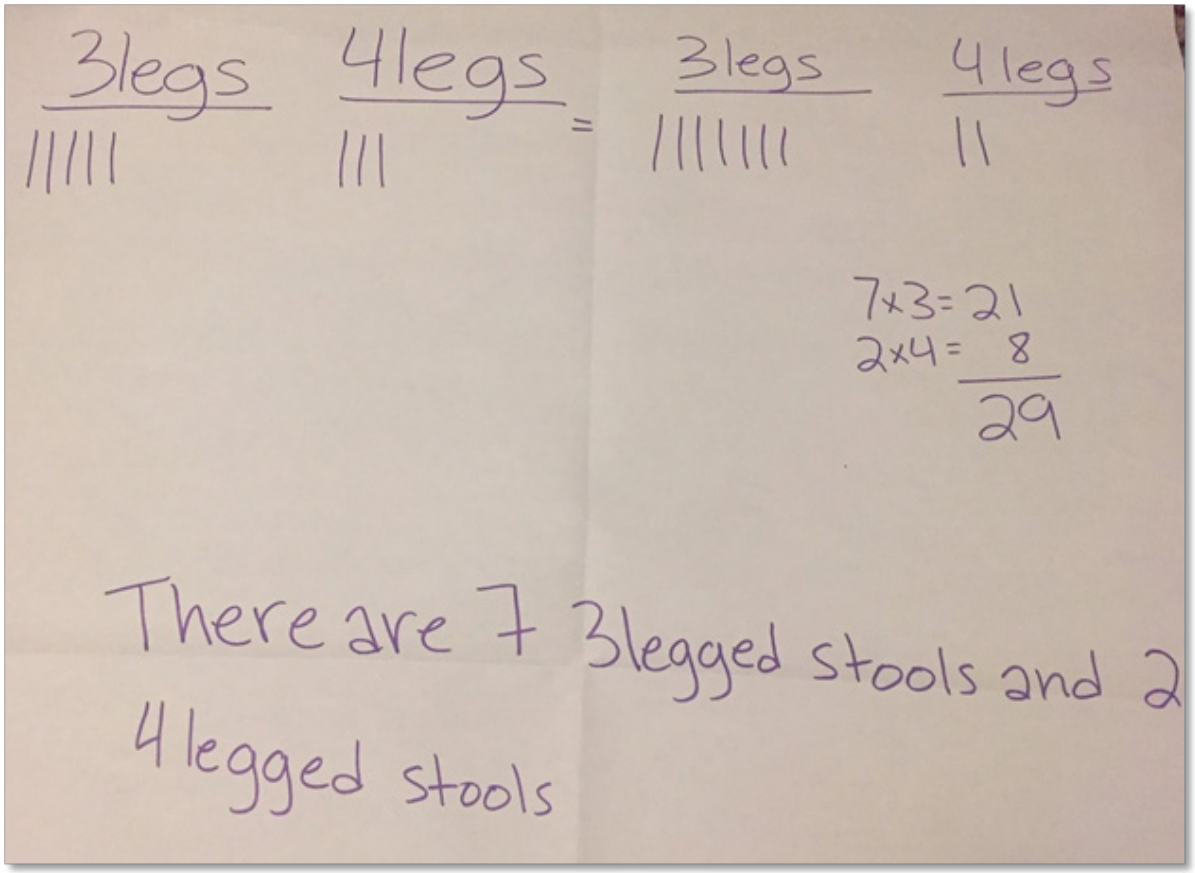
Was there anything else you would have liked to hear this student say?

Handout 5B.3
Proportion Problem II

Directions: Take a few minutes to review the solution to Proportion Problem II.

> Ramona's furniture store has a choice of 3-legged stools and 4-legged stools. There are five more 3-legged stools than 4-legged stools. When you count the legs of the stools, there are exactly 29 legs. How many 3-legged and 4-legged stools are there in the store?

Student Work:



Handout 5B.4

Video Viewing Guide – Proportion Problem II – Focus on Student Explanation

Directions: As you view the video, determine whether the student exhibited any of the behaviors listed in Column 3 (**Did the student…**) while explaining her solution. Check the appropriate column to indicate whether the student's statement was made with or without teacher support. Leave the item blank, if the student did not exhibit the behavior.

Without Support	With Teacher Support	Did the student…
		Did the student state what the problem was about?
		Did the student state what she is trying to find?
		Did the student talk about which information is important for solving the problem?
		Did the student explain her strategy as it relates to key information in the problem?
		Did the student explain why she thinks her answer is reasonable?

Notes about the interaction:

Was there anything else you would have liked to hear this student say?

Handout 5B.5
The Bead Problem

Cindy is making a necklace and using a pattern with her beads. For every 3 jade beads she puts on the necklace, she adds 10 magenta beads. If she has 9 jade beads, how many magenta beads will she need to make the necklace.

Follow up question: How many beads are on the necklace?

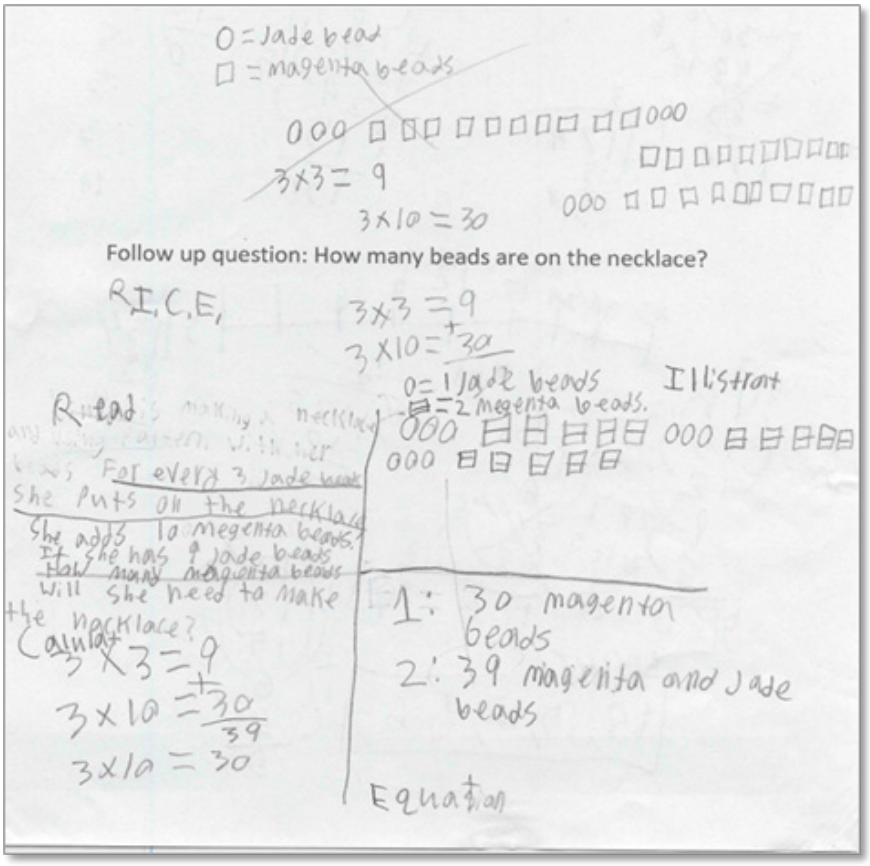
Write anticipatory questions:

1. _____

2. _____

3. _____

Student Work:



Handout 5B.6

The Bead Problem – Categorizing Teacher Comments in the Script

Directions: Identify the type of questions the teacher asks the student by entering the initials for the type of question in the blank line beside the question.

Clarifying Questions (CL): The teacher encourages the student to clarify a statement or approach, examples include "How?" or "What do you mean?" or "So?"

Confirmation Questions (CF): The teacher repeats something the student said in a new way to confirm she understood the student's approach.

Leading Questions (L): The teacher can get the student to think about what *might be their next step* in solving a problem or what *conclusions they could draw* from what they just did and why it did or did not work.

Elaboration or Extended Feedback (E): When a teacher paraphrases and summarizes the student's thinking it is a form of elaborating on what the student said. The teacher's feedback extends the student's answer.

1. _____ **T:** Can you explain some of your thinking and how you went through this problem?

S: *I started off making the pattern, so I could visually see it and I did some multiplication to find the numbers, but then I realized there's a better way of doing it.*

2. _____ **T:** Can you explain how you started making the pattern? What made you make the decision to put circles and the squares and the number of circles and squares you used?

S: *I made a key that says that one circle makes one jade bead and a square equals magenta beads. I did three because it says three jade beads. And I put three circles and then it says she puts on the necklace she adds in magenta beads and so I added ten squares and then I repeated that three times.*

3. _____ **T:** And what made you decide not to continue with this way of solving the problem?

S: *Because there was another question that said how many beads are on the necklace and so I thought because I really didn't know what you called the method, so I thought of a thing we do in school which is called R.I.C.E.*

4. _____ **T:** So down here it looks like you used the R.I.C.E. method and you thought that might be better than what you were doing up here to solve this follow up question? Is that why you chose to shift from solving it this way to solve it this way (points to each of the two different methods)?

5. _____ **T:** Can you explain what you did for your R.I.C.E. method. What were each of these pieces?

S: *The first one is Read so I put the important details and then the second one is for Read I underline for every three jade beads she puts on the necklace she puts on 10 magenta beads b/c I thought that was one of the important parts to know.*

6. _____ **T:** And I noticed that you did that up here too when you were (points to underlined important information the student identified) first reading it in your first method, you also underlined these, so you were highlighting the important information when you were going through it. Very good.

S: *For the second one, its Illustrate, so I did kind of the same pattern, except I made it simpler instead of one square equaling two magenta beads. I did one square with a line equals 2 magenta beads and I just continued with the pattern.*

7. _____ **T:** And did you use this method with one square equaling two magenta beads?

8. _____ **T:** Was that to save you some space as you were working?

S: *Yes.*

T: All right good thinking.

S: *And then for the C is Calculating.*

T: I see right here it says to calculate. Good.

S: *Up here it says 3 jade beads she put on the necklace. On here it says if she has 9 jade beads, how many magenta beads would she need to make so I did 3 x 3 was 9 and then 3 x 10 using the 3 because I used 3 here. I would use the 3 again to multiply it by 10 so I got 30 and then I added that to get 39. And then for the other question of how many beads are on the necklace. (student pauses)*

9. _____ **T:** So, you showed you calculated here. It looks like this—part of R.I.C.E. is the equation. Can you explain how you built up that equation?

S: *So, for the first one-30 magenta beads and for the second one it was 39 magenta beads.*

10. _____ **T:** So, what I'm thinking is that you had 30 magenta beads and you knew that you needed the nine more so is this where you showed that this was 30 and the 39?

S: *Yah*

11. _____ **T:** Can you think of a way you could have gotten to this solution of 39 beads with the information you got here? (points to 000 box box box, etc.)

S: *I could have counted by 10s for this and then I would have gotten to 30 and then I could have just added 9 to it.*

12. _____ **T:** And if you take a look at the work you did here, you're thinking 3 x 3 = 9 and also the 3 x 10 = 30, so you would have arrived at 39 by doing what?

S: *Adding*

T: You've got it very good.

Handout 5B.7
The Cycle Center Problem

The Cycle Center has bicycles and tricycles in the storeroom. There are at least two of each, and there are more bicycles than tricycles. There are 23 wheels altogether. How many bicycles and tricycles are in the storeroom?

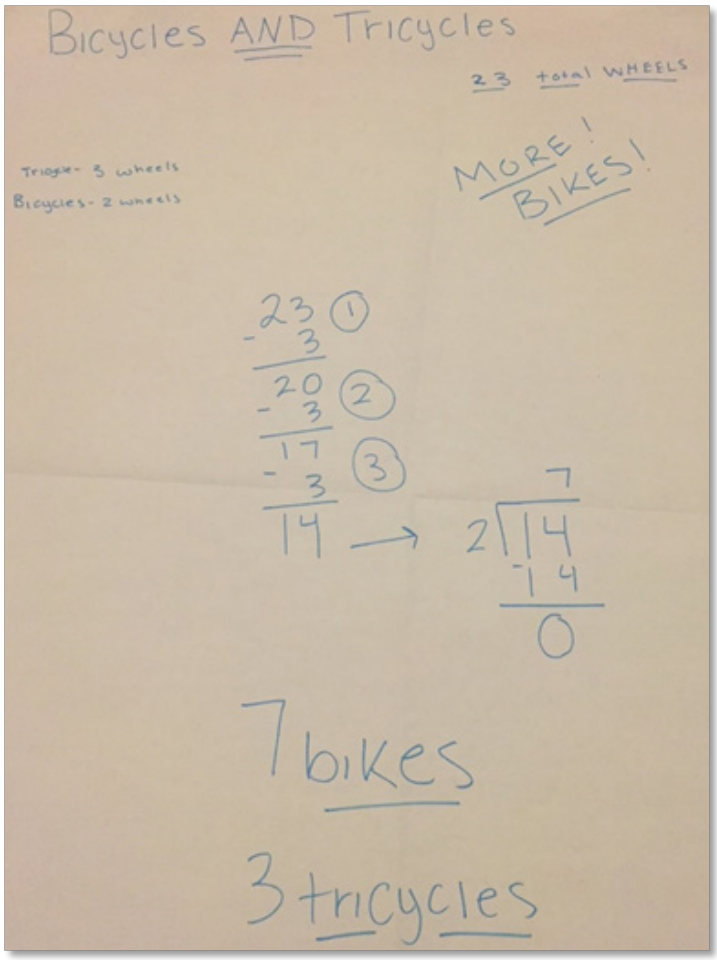
Write anticipatory questions:

1. _____

2. _____

3. _____

Student Work:



Handout 5B.8

The Cycle Center Problem – Categorizing Teacher Comments in Script

Directions: Identify the type of questions the teacher asks the student by entering the initials for the type of question in the blank line beside the question.

Clarifying Questions (CL): The teacher encourages the student to clarify a statement or approach, examples include "How?" or "What do you mean?" or "So?"

Confirmation Questions (CF): The teacher repeats something the student said in a new way to confirm she understood the student's approach.

Leading Questions (L): The teacher can get the student to think about what *might be their next step* in solving a problem or what *conclusions they could draw* from what they just did and why it did or did not work.

Elaboration or Extended Feedback (E): When a teacher paraphrases and summarizes the student's thinking it is a form of elaborating on what the student said. The teacher's feedback extends the student's answer.

1. _____ **T:** Can you say a little about how you solved this problem?

S: *So we started out with 23 total wheels and there's 3 wheels on the tricycle so I started taking away 3 wheels and that led me to 20 wheels and I took away 3 more and that got me to 17 wheels and take away 3 more which is 14 and this is an even number so I can use that so I divide 2 b/c there are 2 wheels on a bicycle which leaves me with 7 and so I have 7 bicycles and 3 tricycles.*

2. _____ **T:** Can you say a little bit more—why didn't you just stop at 1 tricycle. You mentioned that it came down to an even number so often we had one tricycle why didn't you just stop and divide by 2 there?

S: *So, in the problem it said there were at least two of each, two bicycles and tricycles so there would only be one tricycle if I left it at 20.*

3. _____ **T:** Oh, so one of the parameters of the problem was that there had to be at least two of each. Why didn't you stop the second time around?

4. _____ **T:** How did you know not to stop at 2 tricycles?

S: *So, 17 is not an even number. It's an odd number so if I would have divided it by 2, it would be a fraction which you really can't have a fraction of a bike, it needs to be a whole bike.*

T: That is true.

5. _____ **T:** If we were going to extend this kind of pattern you got going, could we get another answer where we still had more bikes than trikes—if we had a least two of each but we had a different answer than this?

S: *Yes. We could because if you subtracted 3 from here (points to 14) that would be 12…no 11 and if you take 3 away from 11 that would be 8 and you could do 8 divided by 2 which is four bikes wait…*

6. _____ **T:** And how many tricycles would that be?

S: *That would be more tricycles so no you could not do that.*

T: Great work and high five!

Handout 5B.9
Reflection

Directions: Complete the items below in preparation for a PLC follow up meeting to discuss how the lesson went.

1. Did the students who explained the solution need a lot of probing questions from the teacher?

2. Did any of your students approach the problem in an unexpected way?

3. Were you able to think of probing questions that worked well with your students' approaches to the problem?

4. How did your students respond to your questioning?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